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PROBLEMS IN THE THEORY OF CONTINUOUS GROUPS.

BY CHAS. L. BOUTON.

1. Find the finite equations of the following G_1's, using, whenever possible, both the method of integrating a simultaneous system, and the series method:

a. $Uf = ayp + bxq$, where a and b are constants.

(Express the result in terms of hyperbolic functions.)

b. $Uf = (x^2 - y^2)\,p + 2xyq$.

c. $Uf = (x + y)^n\,(p - q)$.

d. $$Uf = \rho_\mu(x_1 \cdots x_n) \sum_{i=1}^{i=n} x_i\,p_i,$$

where ρ_μ is a *homogeneous* function of the x's of degree μ. Discuss the case $\mu = 0$.

2. Reduce the transformations given in 1 to canonical form (*Lie*, pp. 55, 298).*

3. Form the alternant† (Klammerausdruck) of the following transformations:

$$U_1 f = \sum_{i=1}^{i=n} x_i p_i\,, \quad U_2 f = \sum_{i=1}^{i=n} \phi_i\, p_i\,,$$

where the ϕ_i's are *homogeneous* functions, all of the *same* degree μ.

4. Form all possible new transformations by combining

$$U_1 f = p\,, \quad U_2 f = x^2 p + xyq\,, \quad U_3 f = xyp + y^2 q\,,$$

and then combining the resulting transformations.

5. Given the r infinitesimal transformations

$$U_k f = \rho_k(x_1 \cdots x_n) \sum_{i=1}^{i=n} x_i p_i\,, \quad (k = 1, 2, \cdots r),$$

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* In this article the text book, *Differentialgleichungen*, by Sophus Lie, edited by Scheffers, will be referred to as *Lie*.

† Cayley, On Reciprocants and Differential Invariants. *Quarterly Journal of Math.*, Vol. XXVI (1893), p. 303.

where the ρ_k's are homogeneous functions of the x's, *all* of the same degree μ. Show that these transformations form a group. Determine in closed form the finite equations of this G_r. Discuss the case $\mu = 0$.

6. Find the family of curves which cut the tangents of the circle $x^2 + y^2 = r^2$ at the constant angle $\tan^{-1} m$. Integrate the resulting differential equation by finding geometrically a G_1 of which it admits, and performing a quadrature. Then use the method of Scheffers (*Lie*, p. 154) to determine a second integrating factor, and so get the equation of the curves *without a quadrature*. As a special case find the equation of the involute of a circle.

Result, in the general case:

$$\tan^{-1} \frac{yr - x\sqrt{x^2 + y^2 - r^2}}{xr + y\sqrt{x^2 + y^2 - r}} - m \log\left[\sqrt{x^2 + y^2 - r^2} - rm\right] = \text{const.}$$

7. An ordinary differential equation of the first order admits of $U_1 f$ and $U_2 f$. If $(U_2 U_1) \equiv 0$, show that $\Omega \equiv$ constant. (*Lie*, p. 130.)

8. Integrate the following differential equations, which define families of isothermal curves:

$a.$ $xy'(x^2 + y^2 - 1) - y(x^2 + y^2 + 1) = 0.$

$b.$ $xy'(x^2 - 3y^2) + y(y^2 - 3x^2) = 0.$

$c.$ $y' \sin x \sinh y - (1 + \cos x \cosh y) = 0.$

9. Determine the types of invariant differential equations, integrating factors, and substitution to separate the variables, for the following groups, where k is a constant, and ξ, η, ϕ, ψ represent arbitrarily given functions:*

$a.$ $Uf = kyp + xq.$

$b.$ $Uf = kxp + yq.$

$c.$ $Uf = \dfrac{1}{\xi(x)}\left[p + \psi'(x) q\right].$

$d.$ $Uf = \eta(y)\left[\psi'(y) p + q\right].$

$e.$ $Uf = \dfrac{1}{\phi'(x)} p - \dfrac{1}{\psi'(y)} q.$

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* This list is a generalization of that given by Page, *Ordinary Differential Equations*, p. 96.

f. $Uf=\rho_\mu(x,y)\left[xp+yq\right]$,

where ρ_μ is a homogeneous function of x and y of degree μ.

g. $Uf=\dfrac{1}{\phi(x,y)}q$.

10. An m times extended transformation being

$$U^{(m)}f=\xi p+\eta q+\sum_{k=1}^{k=m}\eta_k\frac{\partial f}{\partial y^{(k)}},$$

show that

$$\eta_k\equiv\frac{d^k\eta}{dx^k}-\sum_{s=0}^{s=k-1}\frac{d^s}{dx^s}\left(y^{(k-s)}\frac{d\xi}{dx}\right),$$

where the differentiations with respect to x are total.

11. Determine all of the transformations of the following equations (*Lie*, p. 389), and hence simplify the integration of the equations as much as possible:

a. $y''-xyy'=0$.

b. $x^2y''-xy'+y=0$.

c. $xyy''-xy'^2-yy'=0$.

d. $(y-xy')y''-4y'^2=0$.*

e. $3y''y^{\text{iv}}-5y'''^2=0$.

12. Use the method of *Lie*, p. 389, to determine *all* the infinitesimal transformations which transform any circle into a circle.

13. Determine the analytic expression and the geometric meaning of the invariants of the following systems:

In the plane.

a. A right line and a circle, under the group of similitudinous transformations.

b. Two parabolas, under the general linear group.

c. A point, a right line, and a conic, under the general projective group.

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* Forsyth, *Differential Equations*, p. 125.

d. A point and two circles, under the G_6 of all circular transformations. (See 12.)

e. x, y, y', y'', and a right line, under the group of similitudinous transformations.

In space.

f. Four right lines, under the general projective group.

g. Two cones of the second degree, under the same group.

h. Two conics, under the same group.

14. Identify the group determined in 4 with one of those given by Lie, *Continuierliche Gruppen*, p. 288, and find the change of variables which carries that group into the typical one.

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